

# The Cayley plane and String bordism

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**ABSTRACT.** This paper shows that the kernel of the Witten genus tensor  $\mathbf{Z}[\frac{1}{6}]$  is generated by total spaces of Cayley plane bundles, but only after restricting the Witten genus to string bordism. It does so by showing that the divisibility properties of Cayley plane bundle characteristic numbers arising in Borel-Hirzebruch Lie-group-theoretic calculations correspond precisely to the divisibility properties arising in the Hovey-Ravenel-Wilson BP-Hopf-ring-theoretic calculation of string bordism at primes  $> 3$ .

## Introduction

This paper shows that an affinity between bordism rings and projective spaces extends further than previously known.

The first manifestation of this affinity is the fact that *real projective bundles generate the unoriented bordism ring*  $\pi_*\mathbf{MO}$ . In more detail, Thom showed that  $\pi_*\mathbf{MO}$  is a polynomial ring over  $\mathbf{Z}/2$  with one generator in each dimension not of the form  $2^k - 1$ ; Milnor [MS74, Problem 16-F] showed that a smooth degree- $(1, 1)$  hypersurface  $H \hookrightarrow \mathbf{RP}^i \times \mathbf{RP}^j$  can serve as generator if  $1 < i < j$  and if  $\binom{i+j}{i}$  is not divisible by 2 (equivalently, if there are no ‘carries’ when adding  $i$  to  $j$  in base 2). If  $i \leq j$  then the projection  $H \rightarrow \mathbf{RP}^i$  is a fiber bundle with fiber  $\mathbf{RP}^{j-1}$  (see [HBj92, Ch. 4]).

The second manifestation of this affinity is the fact that *complex projective bundles generate the oriented bordism ring*  $\pi_*\mathbf{MSO}$  after inverting 2. In more detail,  $\pi_*\mathbf{MSO}[\frac{1}{2}]$  is a polynomial ring over  $\mathbf{Z}[\frac{1}{2}]$  with one generator in each dimension  $4k$ ; in each such dimension, a  $\mathbf{Z}$ -linear combination of smooth degree- $(1, 1)$  hypersurfaces  $H \hookrightarrow \mathbf{CP}^i \times \mathbf{CP}^j$  can serve as generator. If  $i \leq j$  then the projection  $H \rightarrow \mathbf{CP}^i$  is a fiber bundle with fiber  $\mathbf{CP}^{j-1}$ . We shall return to this manifestation in more detail in the next section.

The third manifestation of this affinity is the fact that *quaternionic projective bundles almost generate the spin bordism ring*  $\pi_*\mathbf{MSpin}$  after inverting 2. In more detail, the forgetful homomorphism  $\pi_*\mathbf{MSpin} \rightarrow \pi_*\mathbf{MSO}$  becomes an isomorphism after inverting 2; Kreck-Stolz [KS93] showed that an  $\mathbf{HP}^2$  bundle can serve as generator in each dimension except 4. ( $\mathbf{HP}^2$  is 8-dimensional, after all.) In fact, Kreck-Stolz tackled the prime 2 as well and showed that  $\mathbf{HP}^2$  bundles generate the kernel of the Atiyah invariant:

$$\alpha : \pi_*\mathbf{MSpin} \rightarrow \pi_*\mathbf{ko} \cong \mathbf{Z}[\eta, \omega, \mu] / (2\eta, \eta^3, \eta\omega, \omega^2 - 4\mu)$$

where  $\eta, \omega, \mu$  have degree 1, 4, 8, respectively. In a sense, then, the Atiyah invariant measures the failure of  $\mathbf{HP}^2$  bundles to generate the spin bordism ring. Kreck-Stolz used this to show that *ko-theory equals spin bordism modulo  $\mathbf{HP}^2$  bundles*.

The fourth manifestation of this affinity is the subject of this paper: *Cayley plane* ( $\mathbf{CaP}^2$ ) bundles—that is, octonionic projective plane bundles—almost generate the string bordism ring  $\pi_*\mathrm{MO}\langle 8 \rangle$  after inverting 6. (The Cayley plane is 16-dimensional so Cayley plane bundles cannot possibly generate the whole string bordism ring.) In fact, we show that:

**Theorem 1.** *Cayley plane bundles generate the kernel of the Witten genus tensor  $\mathbf{Z}[\frac{1}{6}]$ :*

$$\phi_W : \pi_*\mathrm{MO}\langle 8 \rangle[\frac{1}{6}] \rightarrow \pi_*\mathrm{tmf}[\frac{1}{6}] \cong \mathbf{Z}[\frac{1}{6}][\mathbf{G}_4, \mathbf{G}_6]$$

In a sense, then, the Witten genus measures the failure of  $\mathbf{CaP}^2$  bundles to generate the string bordism ring.

An interesting complication here is that Theorem 1 only appears to be true after restricting the Witten genus to string bordism. In other words,  $\mathbf{CaP}^2$  bundles do not appear to generate the kernel of the quasi-modular-form-valued Witten genus  $\pi_*\mathrm{MSO}[\frac{1}{6}] \rightarrow \mathbf{Z}[\frac{1}{6}][\mathbf{G}_2, \mathbf{G}_4, \mathbf{G}_6]$ . Far from it, in fact: the subring of  $\pi_*\mathrm{MSO}[\frac{1}{6}]$  generated by total spaces of oriented  $\mathbf{CaP}^2$  bundles (and string manifolds of dimension  $< 16$ ) appears to coincide with the image of the forgetful homomorphism  $\pi_*\mathrm{MO}\langle 8 \rangle[\frac{1}{6}] \rightarrow \pi_*\mathrm{MSO}[\frac{1}{6}]$ . As we shall see, this homomorphism is the inclusion of an intricate, non-polynomial subring.

It is already known that  $\mathbf{CaP}^2$  bundles generate the kernel of the Witten genus tensor  $\mathbf{Q}$ . (See p. 12 for a history of this result.) But since stable rational homotopy theory is trivial, rational results are unsatisfying to homotopy theorists. This paper does not tackle the primes 2 or 3, the primes at which  $\mathrm{tmf}$  is most interesting. But the author hopes that homotopy theorists will be pleased to see geometry brought to bear on the primes  $> 3$ . As far as the author knows, this paper gives the first geometrically explicit list of generators for  $\pi_*\mathrm{MO}\langle 8 \rangle[\frac{1}{6}]$ .

Note that the spectrum  $\mathrm{tmf}[\frac{1}{6}]$  is not isomorphic to the quotient spectrum  $\mathrm{MO}\langle 8 \rangle[\frac{1}{6}]/I$  for any ideal  $I$  of  $\pi_*\mathrm{MO}\langle 8 \rangle[\frac{1}{6}]$ . In fact, the coefficient rings  $\pi_*\mathrm{tmf}[\frac{1}{6}]$  and  $\pi_*(\mathrm{MO}\langle 8 \rangle[\frac{1}{6}]/I)$  are not isomorphic for any ideal  $I$  of  $\pi_*\mathrm{MO}\langle 8 \rangle[\frac{1}{6}]$ . This is because  $\ker(\phi_W)$  is not generated by a regular sequence. Indeed,  $\mathrm{MO}\langle 8 \rangle[\frac{1}{6}]/\ker(\phi_W)$  will have  $p$ -torsion for all  $p > 3$ , regardless which sequence of generators one uses to construct it.

Throughout this paper the italic letter  $p$  will denote a prime number. The Roman letter  $p$  will denote the Pontrjagin class.

## 1. Pontrjagin numbers and oriented bordism

This section briefly reviews background material on Pontrjagin classes and the oriented bordism ring. This serves both to fix notation as well as to illustrate how the results of this paper extend well-known calculations.

The  $i$ th Pontrjagin class of a real vector bundle  $V$  is by definition  $p_i(V) = (-1)^i c_{2i}(V \otimes \mathbf{C})$ . It pulls back from the universal  $i$ th Pontrjagin class  $p_i$  in  $H^*(\mathrm{BO}(4n), \mathbf{Z})$  for  $n \geq i$ , which in turn may be identified with the  $i$ th elementary symmetric polynomial. This is because the  $i$ th Pontrjagin class of a sum of complex line bundles is the  $i$ th elementary symmetric polynomial in the first Pontrjagin classes of the individual line bundles,  $p(L_1 \oplus \cdots \oplus L_n) = \prod(1 + p_1(L_i))$ . (The driving force behind this is the fact that, in ordinary cohomology, the total Chern class is logarithmic,  $c(V_1 \oplus V_2) = c(V_1) \cdot c(V_2)$ .)

It is a basic fact that the ring of symmetric polynomials is a polynomial ring on the elementary symmetric polynomials. There are other symmetric polynomials of geometric interest, though. Given a partition  $I = i_1, \dots, i_r$  let  $s_I$  denote the polynomial  $\sum p_1(L_1)^{i_1} \cdots p_1(L_r)^{i_r}$  where the sum runs over all distinct monomials obtained by permuting  $L_1, \dots, L_n$ . Each  $s_I$

is a symmetric polynomial, so may be written as a polynomial in the elementary symmetric polynomials. Thus we may associate to each  $s_I$  a polynomial in the Pontrjagin classes, which we also denote  $s_I$ . Note in particular that  $s_1, s_{1,1}, s_{1,1,1}, \dots$  are the Pontrjagin classes  $p_1, p_2, p_3, \dots$  themselves. The geometric significance of the classes  $s_I$  comes from the following lemma (Lemma 16.2 of [MS74]).

**Lemma (Thom).** *If  $0 \rightarrow V_1 \rightarrow W \rightarrow V_2 \rightarrow 0$  is an exact sequence of vector bundles then:*

$$s_I(W) = \sum_{JK=I} s_J(V_1) s_K(V_2)$$

where the sum ranges over all partitions  $J$  and  $K$  with juxtaposition  $JK$  equal to  $I$ .

This implies that  $s_n$  of the tangent bundle of a nontrivial product of closed oriented manifolds vanishes. Since Pontrjagin numbers detect equality in  $\pi_*\text{MSO}[\frac{1}{2}]$ , it follows that a closed oriented manifold  $M^{4n}$  is decomposable in  $\pi_*\text{MSO}[\frac{1}{2}]$  iff the number  $s_n[M^{4n}] := \int_M s_n(TM)$  vanishes. (The integral  $\int_M$  here denotes the pushforward to a point  $H^{4n}(M) \rightarrow H^0(\text{pt}) \cong \mathbb{Z}$ .) Since  $\pi_*\text{MSO} \otimes \mathbb{Q}$  is a polynomial ring over  $\mathbb{Q}$  with one generator in each dimension  $4n > 0$ , a sequence  $M^4, M^8, M^{12}, \dots$  therefore generates  $\pi_*\text{MSO} \otimes \mathbb{Q}$  iff  $s_n[M^{4n}] \neq 0$  for each  $n \geq 1$ . As mentioned in the introduction, however, inverting just the prime 2 is enough to make  $\pi_*\text{MSO}$  a polynomial ring. It follows that the numbers  $s_n$  suffice to recognize a sequence of generators for  $\pi_*\text{MSO}[\frac{1}{2}]$ , but it turns out that these numbers have unexpected divisibility properties.

For any integer  $n$  and any prime  $p$  let  $\text{ord}_p(n)$  denote the  $p$ -adic order of  $n$ , that is, the largest integer  $v$  such that  $p^v$  divides  $n$ .

**Theorem** (see [Sto68, p. 180]). *A sequence  $M^4, M^8, M^{12}, \dots$  generates  $\pi_*\text{MSO}[\frac{1}{2}]$  iff:*

- For any integer  $n > 0$  and any odd prime  $p$ :

$$\text{ord}_p(s_n[M^{4n}]) = \begin{cases} 1 & \text{if } 2n = p^i - 1 \text{ for some integer } i > 0 \\ 0 & \text{otherwise} \end{cases}$$

Equivalently, if  $p$  is odd then the Hurewicz homomorphism  $\pi_*\text{MSO}_{(p)} \rightarrow H_*\text{MSO}_{(p)}$ , after passing to indecomposable quotients, is multiplication by  $\pm p$  in degrees of the form  $2(p^i - 1)$  and is an isomorphism otherwise. (See [Rav86, Theorem 3.1.5] where the special behavior in degrees  $2(p^i - 1)$  ultimately comes from the degrees of the generators  $v_i$  of  $\pi_*\text{BP}$ .)

Now we return to the second manifestation of the affinity discussed in the introduction.

**Proposition.** *If  $H \hookrightarrow \mathbb{CP}^i \times \mathbb{CP}^{2n-i+1}$  is a smooth complex hypersurface of degree  $(1, 1)$  and  $1 < i < 2n$  then:*

$$s_n[H] = -\binom{2n+1}{i}$$

**PROOF.** Since the tangent bundle of the ambient manifold  $\mathbb{CP}^i \times \mathbb{CP}^{2n-i+1}$  splits non-trivially, the lemma of Thom above implies that  $s_n(\text{TH}) = -s_n(\text{NH})$  where the normal bundle  $\text{NH}$  is isomorphic to the complex line bundle  $\mathcal{O}(1, 1)|_H$ . Since for a complex line bundle  $p_1 = c_1^2$  and since in ordinary cohomology  $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$ , it follows that:

$$s_n(\mathcal{O}(1, 1)) = p_1(\mathcal{O}(1, 1))^n = c_1(\mathcal{O}(1, 1))^{2n} = (x_1 + x_2)^{2n}$$

$$\begin{aligned} s_n[H] &= - \int_H s_n(\mathcal{O}(1,1)|_H) = - \int_H (x_1 + x_2)^{2n} \Big|_H \\ &= - \int_{\mathbf{CP}^i \times \mathbf{CP}^{2n-i+1}} (x_1 + x_2)^{2n+1} = - \binom{2n+1}{i} \quad \square \end{aligned}$$

**Lemma.** *For any integer  $n > 0$  and any odd prime  $p$ :*

$$\text{ord}_p \left[ \text{GCD}_{1 \leq i < 2n} \binom{2n+1}{i} \right] = \begin{cases} 1 & \text{if } 2n+1 = p^i \text{ for some integer } i > 0 \\ 0 & \text{otherwise} \end{cases}$$

In short, then, the divisibility properties of  $s_n$  for oriented manifolds, deduced from homotopy theory, align perfectly with the divisibility properties of  $s_n$  for  $\mathbf{CP}^n$  bundles, deduced from divisibility properties of binomial coefficients.

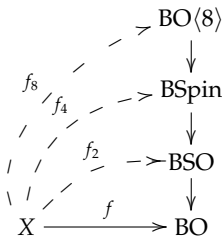
This paper will follow the same outline. First we will deduce the divisibility properties of  $s_n$  (and  $s_{n_1, n_2}$ ) for string manifolds from known results in homotopy theory. Then we will show that these divisibility properties align perfectly with the divisibility properties of  $s_n$  (and  $s_{n_1, n_2}$ ) for Cayley plane bundles, which we will in turn deduce from divisibility properties of binomial coefficients. The arguments and calculations will at each stage be more complicated than for oriented bordism and complex projective bundles, but the outline and spirit will be the same.

## 2. How to recognize generators for string bordism

In the preceding section we stated a criterion, involving the number  $s_n$ , which ensures that a sequence  $M^4, M^8, M^{12}, \dots$  generates  $\pi_*\mathrm{MSO}[\frac{1}{2}]$ . The purpose of this section is to establish an analogous criterion (Theorem 2) for the string bordism ring  $\pi_*\mathrm{MO}\langle 8 \rangle[\frac{1}{6}]$ . It turns out that Pontrjagin numbers still suffice to distinguish elements of  $\pi_*\mathrm{MO}\langle 8 \rangle[\frac{1}{6}]$  but, since this ring is not a polynomial ring, the numbers  $s_n$  do not suffice to recognize generators; certain numbers of the form  $s_{n_1, n_2}$  are also needed. As we shall see, the criterion is a consequence of Hovey’s calculation [Hov08] of  $\pi_*\mathrm{MO}\langle 8 \rangle_{(p)}$  for  $p > 3$ .

First recall what string bordism is. Any real vector bundle  $V \rightarrow X$  of rank  $k$  pulls back from the universal rank- $k$  bundle over the classifying space  $\mathrm{BO}(k)$  by a map  $f : X \rightarrow \mathrm{BO}(k)$ .

- An *orientation* of  $V$  is a (homotopy class of) lift  $f_2$  of  $f$  to the 1-connected cover  $\mathrm{BSO} \rightarrow \mathrm{BO}$ . Such a lift exists iff the generator  $w_1$  of  $H^1(\mathrm{BO}, \mathbb{Z}/2)$  pulls back to 0 in  $H^1(X, \mathbb{Z}/2)$ .
  - A *spin structure* on  $V$  is a (homotopy class of) lift  $f_4$  of  $f_2$  to the 3-connected cover  $\mathrm{BSpin} \rightarrow \mathrm{BSO}$ . Such a lift exists iff the generator  $w_2$  of  $H^2(\mathrm{BSO}, \mathbb{Z}/2)$  pulls back to 0 in  $H^2(X, \mathbb{Z}/2)$ .
  - A *string structure* on  $V$  is a (homotopy class of) lift  $f_8$  of  $f_4$  to the 7-connected cover  $\mathrm{BO}\langle 8 \rangle \rightarrow \mathrm{BSpin}$ . Such a lift exists iff the generator  $\frac{1}{2}p_1$  of  $H^4(\mathrm{BSpin}, \mathbb{Z})$  pulls back to 0 in  $H^4(X, \mathbb{Z})$ .
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The bordism spectrum of string manifolds  $\mathrm{MO}\langle 8 \rangle$  is the Thom spectrum of the map  $\mathrm{BO}\langle 8 \rangle \rightarrow \mathrm{BO}$ . Its coefficient ring  $\pi_*\mathrm{MO}\langle 8 \rangle$  is the bordism ring of manifolds equipped with a string structure on their stable normal bundle.

**Theorem 2.** *A set  $S$  generates  $\pi_*\mathrm{MO}\langle 8 \rangle[\frac{1}{6}]$  if:*

- (1) *For each integer  $n > 1$ , there is an element  $M^{4n}$  of  $S$  such that for any prime  $p > 3$ :*

$$\mathrm{ord}_p(s_n[M^{4n}]) = \begin{cases} 1 & \text{if } 2n = p^i - 1 \text{ or } 2n = p^i + p^j \text{ for some integers } 0 \leq i \leq j \\ 0 & \text{otherwise} \end{cases}$$

- (2) *For each prime  $p > 3$  and each pair of integers  $0 < i < j$ , there is an element  $N^{2(p^i + p^j)}$  of  $S$  such that:*

$$s_{(p^i + p^j)/2}[N^{2(p^i + p^j)}] = 0$$

but:

$$s_{(p^i + 1)/2, (p^j - 1)/2}[N^{2(p^i + p^j)}] \not\equiv 0 \pmod{p^2}$$

**Proposition.** *The forgetful homomorphism:*

$$\pi_*\mathrm{MO}\langle 8 \rangle[\frac{1}{6}] \rightarrow \pi_*\mathrm{MSpin}[\frac{1}{6}]$$

*is injective.*

PROOF. It is injective tensor  $\mathbf{Q}$  so its kernel is torsion (since  $\mathbf{Q}$  is a flat  $\mathbf{Z}$ -module). Giambalvo, however, showed that  $\pi_*\mathrm{MO}\langle 8 \rangle$  has no  $p$ -torsion for  $p > 3$  [Gia71, Theorem 4.3].  $\square$

Since Pontrjagin numbers detect equality in  $\pi_*\mathrm{MSpin}[\frac{1}{2}] \cong \pi_*\mathrm{MSO}[\frac{1}{2}]$  it follows that:

**Corollary.** *Pontrjagin numbers detect equality in  $\pi_*\mathrm{MO}\langle 8 \rangle[\frac{1}{6}]$ .*

To prove Theorem 2 it therefore suffices to determine the image of  $\pi_*\mathrm{MO}\langle 8 \rangle[\frac{1}{6}] \rightarrow \pi_*\mathrm{MSpin}[\frac{1}{6}]$  or, equivalently, to determine the image of  $\pi_*\mathrm{MO}\langle 8 \rangle_{(p)} \rightarrow \pi_*\mathrm{MSpin}_{(p)}$  for each prime  $p > 3$ . The Hovey-Ravenel-Wilson approach [RW74, HR95] to  $\mathrm{BO}4k$  reduces  $\pi_*\mathrm{MO}\langle 8 \rangle_{(p)} \rightarrow \pi_*\mathrm{MSpin}_{(p)}$  to the homomorphism  $\mathrm{BP}_*\mathrm{BP}\langle 1 \rangle_{2(p+1)} \rightarrow \mathrm{BP}_*\mathrm{BP}\langle 1 \rangle_4$ , and Hovey's description [Hov08] of these rings reveals enough information about the image to prove Theorem 2. What follows is a brief summary of the results of [RW74, HR95, Hov08] needed to prove Theorem 2.

First some standard notation. Let  $\mathrm{BP}$  denote the Brown-Petersen spectrum [BP66] with coefficient ring  $\pi_*\mathrm{BP} \cong \mathbf{Z}_{(p)}[v_1, v_2, \dots]$  where  $\deg(v_i) = 2(p^i - 1)$ . Let  $\mathrm{BP}\langle 1 \rangle$  denote the Johnson-Wilson spectrum [JW73] obtained from  $\mathrm{BP}$  by killing the ideal  $(v_2, v_3, \dots)$  of  $\pi_*\mathrm{BP}$ ; its coefficient ring is  $\pi_*\mathrm{BP}\langle 1 \rangle \cong \mathbf{Z}_{(p)}[v_1]$ . The infinite loop space obtained by applying the  $k$ -th space functor to a spectrum  $X$  will be denoted  $X_k$ .

Recall that the ring homomorphism  $\pi_*\mathrm{BP}\langle 1 \rangle \rightarrow \pi_*\mathrm{ku}_{(p)}$  taking  $v_1$  to  $v^{p-1}$  lets one identify  $\pi_*\mathrm{ku}_{(p)} \cong \mathbf{Z}_{(p)}[v]$  with  $\pi_*\mathrm{BP}\langle 1 \rangle[v]/(v_1 - v^{p-1})$ . This identification extends to a multiplicative splitting of spectra:

$$\mathrm{ku}_{(p)} \cong \prod_{i=1}^{p-2} \Sigma^{2i} \mathrm{BP}\langle 1 \rangle$$

Multiplication by  $v$  on the left corresponds to the (upward) shift of factors on the right, the shift from top to bottom factor being accompanied by multiplication by  $v_1$ .

Since, for  $k$  even,  $\mathrm{BU}\langle k \rangle$  can be taken as the  $k$ -th space of  $\mathrm{ku}$ , this implies that there is a  $p$ -local decomposition of  $H$ -spaces:

$$\mathrm{BU}\langle k \rangle_{(p)} \cong \prod_{i=1}^{p-2} \mathrm{BP}\langle 1 \rangle_{k+2i}$$

There is an analogous splitting of  $\mathrm{BO}\langle k \rangle_{(p)}$  for  $p > 2$ .

**Theorem** ([HR95, Corollary 1.5]). *If  $k$  is divisible by 4 and  $p > 2$  then there is a  $p$ -local decomposition of  $H$ -spaces:*

$$\mathrm{BO}\langle k \rangle_{(p)} \cong \prod_{i=0}^{(p-3)/2} \mathrm{BP}\langle 1 \rangle_{k+4i}$$

Under this decomposition the map  $\mathrm{BO}\langle k+4 \rangle \rightarrow \mathrm{BO}\langle k \rangle$  corresponds to the identity map on the factors  $\mathrm{BP}\langle 1 \rangle_{k+4i}$  for  $0 < i < \frac{1}{2}(p-3)$  and to  $[v_1] : \mathrm{BP}\langle 1 \rangle_{k+2p-2} \rightarrow \mathrm{BP}\langle 1 \rangle_k$  on the remaining factor.

If  $k = 4$  then the situation looks like this:

$$\begin{array}{ccccccc} \mathrm{BO}\langle 8 \rangle_{(p)} & \cong & \mathrm{BP}\langle 1 \rangle_8 & \times & \mathrm{BP}\langle 1 \rangle_{12} & \times & \cdots & \cdots & \times & \mathrm{BP}\langle 1 \rangle_{2p+2} \\ \downarrow & & \swarrow & & \searrow & & \cdots & & \swarrow & \searrow \\ \mathrm{BO}\langle 4 \rangle_{(p)} & \cong & \mathrm{BP}\langle 1 \rangle_4 & \times & \mathrm{BP}\langle 1 \rangle_8 & \times & \mathrm{BP}\langle 1 \rangle_{12} & \times & \cdots & \times & \mathrm{BP}\langle 1 \rangle_{2p-2} \end{array}$$

Hovey shows that  $\pi_*\mathrm{MO}\langle 8 \rangle_{(p)}$  is isomorphic as a ring to a quotient of the  $\mathrm{BP}$ -homology of this splitting. To state his result precisely, we need to introduce some notation. If  $p > 2$  then there is a natural map  $\mathrm{MO}\langle 8 \rangle \rightarrow \mathrm{MSO} \rightarrow \mathrm{BP}$ . If  $p > 3$  then the induced homomorphism  $\mathrm{BP}_*\mathrm{MO}\langle 8 \rangle \rightarrow \mathrm{BP}_*\mathrm{BP}$  is surjective [Hov08, Lemma 2.1]. For each positive integer  $i$ , choose a generator  $u_i$  in  $\mathrm{BP}_{2(p^i-1)}\mathrm{MO}\langle 8 \rangle$  mapping to the generator  $t_i$  of  $\mathrm{BP}_*\mathrm{BP} \cong \mathrm{BP}_*[t_1, t_2, \dots]$ .

**Theorem** ([Hov08, Theorem 2.4]). *If  $p > 3$  then:*

$$\begin{aligned} \pi_*\mathrm{MO}\langle 8 \rangle_{(p)} &\cong \mathrm{BP}_*\mathrm{MO}\langle 8 \rangle / (u_1, u_2, \dots) \\ &\cong \mathrm{BP}_*\mathrm{BP}\langle 1 \rangle_8 \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP}\langle 1 \rangle_{12} \\ &\quad \otimes_{\mathrm{BP}_*} \cdots \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP}\langle 1 \rangle_{2p-2} / (u_1, u_2, \dots) \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP}\langle 1 \rangle_{2p+2} \\ \pi_*\mathrm{MSpin}_{(p)} &\cong \mathrm{BP}_*\mathrm{BP}\langle 1 \rangle_4 \otimes_{\mathrm{BP}_*} \cdots \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP}\langle 1 \rangle_{2p-2} / (u_1, u_2, \dots) \end{aligned}$$

So to understand the forgetful homomorphism  $\pi_*\mathrm{MO}\langle 8 \rangle_{(p)} \rightarrow \pi_*\mathrm{MSpin}_{(p)}$ , it suffices to understand the homomorphism  $[v_1]_* : \mathrm{BP}_*\mathrm{BP}\langle 1 \rangle_{2p+2} \rightarrow \mathrm{BP}_*\mathrm{BP}\langle 1 \rangle_4$ . As we shall see, it is the inclusion into a polynomial ring of a non-polynomial subring. A toy model worth bearing in mind is the inclusion  $\mathbb{Z}[5x, y, xy] \hookrightarrow \mathbb{Z}[x, y]$ .

Instead of studying each ring  $\mathrm{BP}_*\mathrm{BP}\langle 1 \rangle_n$  individually, Hovey exploits the fact that they fit together to form a Hopf ring  $\mathrm{BP}_*\mathrm{BP}\langle 1 \rangle_*$ . In particular there is a circle product:

$$\circ : \mathrm{BP}_*\mathrm{BP}\langle 1 \rangle_m \otimes \mathrm{BP}_*\mathrm{BP}\langle 1 \rangle_n \rightarrow \mathrm{BP}_*\mathrm{BP}\langle 1 \rangle_{m+n}$$

corresponding to the ring spectrum structure of  $\mathrm{BP}\langle 1 \rangle$ . It gives an inductive way to construct elements in the increasingly complicated rings  $\mathrm{BP}_*\mathrm{BP}\langle 1 \rangle_{n+m}$ . In fact, all the elements

we will need can be constructed that way from just two kinds of elements,  $b_{(i)}$  and  $[v_1^i]$ , defined as follows. The complex orientation gives a map  $\mathbf{CP}^\infty \rightarrow \mathbf{BP}\langle 1 \rangle_2$ . Let  $b_i \in \mathbf{BP}_{2i}\mathbf{BP}\langle 1 \rangle_2$  be the image under this map of the BP-homology generator of degree  $2i$ . Let  $b_{(i)}$  denote the generator  $b_{p^i}$  (generators not of this form are decomposable). The homotopy class  $v_1^i$  is represented by a map  $S^0 \rightarrow \mathbf{BP}\langle 1 \rangle_{-2i(p-1)}$ . Let  $[v_1^i] \in \mathbf{BP}_0\mathbf{BP}\langle 1 \rangle_{-2i(p-1)}$  denote the image under this map of the BP-homology generator.

**Theorem** ([Hov08, Theorem 1.2]). *If  $n < 2p$  then  $\mathbf{BP}_*\mathbf{BP}\langle 1 \rangle_n$  is a polynomial algebra over  $\mathbf{BP}_*$  with one generator in each positive even degree congruent to  $n \bmod 2p - 2$ . In a degree  $2m$  of that form, one can take as generator:*

$$x_{2m} = [v_1^i] \circ b_{(0)}^{\circ j_0} \circ b_{(1)}^{\circ j_1} \circ \cdots \circ b_{(k)}^{\circ j_k}$$

where  $m = \sum j_l p^l$  is the  $p$ -adic expansion and  $i = \frac{1}{p-1}(\alpha(m) - \frac{1}{2}n)$  with  $\alpha(m) = \sum j_l$ .

If  $n = 2p + 2$  then  $\mathbf{BP}_*\mathbf{BP}\langle 1 \rangle_n$  is not a polynomial ring over  $\mathbf{BP}_*$ . It has a generator in each degree congruent to  $4 \bmod 2p - 2$  (and greater than 4) but it has two generators in some of these dimensions, and these generators satisfy a relation. Specifically:

- In each degree  $4p^i$  for  $i > 0$  there is one generator:

$$w_{4p^i} = b_{(i)} \circ b_{(i-1)}^{\circ p}$$

- In each degree  $2(p^i + p^j)$  for  $0 \leq i < j$  there is a generator:

$$y_{2(p^i+p^j)} = b_{(i)} \circ b_{(j-1)}^{\circ p}$$

- In each degree  $2(p^i + p^j)$  for  $0 < i < j$  there is a second generator:

$$z_{2(p^i+p^j)} = b_{(i-1)}^{\circ p} \circ b_{(j)}$$

To simplify formulas later on, let  $z_{2(1+p^j)} = 0$  for  $j > 0$ .

- In each of the other degrees—that is, in each degree  $2m$  congruent to  $4 \bmod 2p - 2$  but not of the form  $2(p^i + p^j)$  for any  $0 \leq i \leq j$ —there is a single generator of the form  $x_{2m}$ , defined as in the preceding theorem.

Hovey constructs, for each  $0 < i < j$ , a relation  $r_{ij}$  involving  $y_{2(p^i+p^j)}$ ,  $z_{2(p^i+p^j)}$  and  $p$ . To express it, let  $I$  be the ideal of  $\mathbf{BP}_*$  generated by  $(p, v_1, v_2, \dots)$  and let  $I(n)$  be the kernel of  $\mathbf{BP}_*\mathbf{BP}\langle 1 \rangle_n \rightarrow \mathbf{BP}_*$ .

**Proposition** ([Hov08, Corollary 1.6]). *For any pair of integers  $0 < i < j$  there is a relation in  $\mathbf{BP}_*\mathbf{BP}\langle 1 \rangle_{2p+2}$  of the form:*

$$\begin{aligned} p(z_{2(p^i+p^j)} - y_{2(p^i+p^j)}) &\equiv v_j y_{2(1+p^i)} - v_i \cdot y_{2(1+p^j)} + y_{2(p^{i-1}+p^{j-1})}^p - z_{2(p^{i-1}+p^{j-1})}^p \\ &\quad \bmod I^2 \cdot I(2p+2) + I \cdot I(2p+2)^{*2} + I(2p+2)^{*p+1} \end{aligned}$$

Considering each of these relations as an element  $r_{ij}$  of the  $\mathbf{BP}_*$ -polynomial ring  $R$  on all the generators  $w_{4p^i}, y_{2(1+p^i)}, y_{2(p^i+p^j)}, z_{2(p^i+p^j)}, x_{2m}$  for  $0 < i < j$  and  $2m$  of the form described above, Hovey shows that:

**Theorem** ([Hov08, Theorem 1.7]).

$$R/(r_{ij} \mid 0 < i < j) \rightarrow \mathbf{BP}_*\mathbf{BP}\langle 1 \rangle_{2p+2}$$

is an isomorphism.

Remember that we want to understand the homomorphism  $[v_1]_* : \mathbf{BP}_* \mathbf{BP}\langle 1 \rangle_{2p+2} \rightarrow \mathbf{BP}_* \mathbf{BP}\langle 1 \rangle_4$ . If  $0 < i < j$  then by definition:

$$\begin{cases} [v_1]_* w_{4p^i} &= [v_1] \circ b_{(i)} \circ b_{(i-1)}^{\circ p} \\ [v_1]_* y_{2(1+p^i)} &= [v_1] \circ b_{(0)} \circ b_{(i-1)}^{\circ p} \\ [v_1]_* y_{2(p^i+p^j)} &= [v_1] \circ b_{(i)} \circ b_{(j-1)}^{\circ p} \\ [v_1]_* z_{2(p^i+p^j)} &= [v_1] \circ b_{(i-1)}^{\circ p} \circ b_{(j)} \\ [v_1]_* x_{2m} &= \underbrace{[v_1] \circ [v_1^i]}_{=[v_1^{i+1}]} \circ b_{(0)}^{\circ j_0} \circ b_{(1)}^{\circ j_1} \circ \cdots \circ b_{(k)}^{\circ j_k} \end{cases}$$

Recall that the exponent  $i$  of  $v_1$  appearing in the generator  $x_{2m}$  depends on both  $m$  and  $n$ , specifically  $i = i(m, n) = \frac{1}{p-1}(\alpha(m) - \frac{1}{2}n)$ . So  $i(m, 4) = i(m, 2p+2) + 1$  and the homomorphism carries each generator of  $\mathbf{BP}_* \mathbf{BP}\langle 1 \rangle_{2p+2}$  of the form  $x_{2m}$  to the corresponding generator  $x_{2m}$  of  $\mathbf{BP}_* \mathbf{BP}\langle 1 \rangle_4$ . To relate the images of the other generators to the generators  $x_{2m}$  of  $\mathbf{BP}_* \mathbf{BP}\langle 1 \rangle_4$ , we rely on the following proposition.

**Proposition** ([Hov08, Corollary 1.5]). *For each integer  $i > 0$  there is a relation in  $\mathbf{BP}_* \mathbf{BP}\langle 1 \rangle_2$  of the form:*

$$[v_1] \circ b_{(i-1)}^{\circ p} \equiv v_i \cdot b_{(0)} - p \cdot b_{(i)} - b_{(i-1)}^{*p} \pmod{I^2 \cdot I(2) + I \cdot I(2)^{*2} + I(2)^{*p+1}}$$

If we  $\circ$ -multiply this relation by  $b_{(j)}$  then we obtain a relation in  $\mathbf{BP}_* \mathbf{BP}\langle 1 \rangle_4$ :

$$\begin{aligned} [v_1] \circ b_{(i-1)}^{\circ p} \circ b_{(j)} &\equiv v_i \cdot b_{(0)} \circ b_{(j)} - p \cdot b_{(i)} \circ b_{(j)} - \underbrace{b_{(i-1)}^{*p} \circ b_{(j)}}_{=(b_{(i-1)} \circ b_{(j-1)})^{*p}} \\ &\pmod{I^2 \cdot I(4) + I \cdot I(4)^{*2} + I(4)^{*p+1}} \end{aligned}$$

The bracketed equality is a consequence of the Hopf ring distributive law (see the discussion just before Lemma 1.7 of [HR95]). If  $j = 0$  then (as that discussion points out) the bracketed quantity equals 0. The fact that  $\mathbf{BP}_* \mathbf{BP}\langle 1 \rangle_m \circ I(n)^{*k} \subseteq I(n+m)^{*k}$  is also a consequence of the Hopf ring distributive law.

Substituting  $(i, j) \mapsto (i, i), (1, i), (j, i), (i, j)$  (and subtracting) produces, for  $0 < i < j$ , the following congruences mod  $I^2 \cdot I(4) + I \cdot I(4)^{*2} + I(4)^{*p+1}$ :

$$\begin{cases} [v_1]_* w_{4p^i} &\equiv v_i \cdot x_{2(1+p^i)} - p \cdot x_{4p^i} - x_{4p^{i-1}}^p \\ [v_1]_* y_{2(1+p^i)} &\equiv v_i \cdot x_4 - p \cdot x_{2(1+p^i)} \\ [v_1]_* y_{2(p^i+p^j)} &\equiv v_j \cdot x_{2(1+p^i)} - p \cdot x_{2(p^i+p^j)} - x_{2(p^{i-1}+p^{j-1})}^p \\ [v_1]_* (z_{2(p^i+p^j)} - y_{2(p^i+p^j)}) &\equiv v_i \cdot x_{2(1+p^j)} - v_j \cdot x_{2(1+p^i)} \end{cases}$$

The generator  $v_i$  of  $\mathbf{BP}_*$  is indecomposable of degree- $2(p^i - 1)$  in:

$$\pi_* \mathbf{MSpin}_{(p)} \cong \mathbf{BP}_* \mathbf{BP}\langle 1 \rangle_4 \otimes_{\mathbf{BP}_*} \cdots \otimes_{\mathbf{BP}_*} \mathbf{BP}_* \mathbf{BP}\langle 1 \rangle_{2p-2} / (u_1, u_2, \dots)$$

considered as a  $\mathbf{Z}_{(p)}$ -algebra. So, by the theorem stated in §1,  $p$  divides  $s_{(p^i-1)/2}[v_i]$  to order 1 but does not divide  $s_{m/2}[x_{2m}]$ .



Since all numbers of the form  $s_n$  and  $s_{n_1, n_2}$  vanish on the ideal  $I^2 \cdot I(4) + I \cdot I(4)^{*2} + I(4)^{*p+1}$ , it follows that  $p$  divides:

$$\begin{aligned} & s_{p^i}([v_1] * w_{4^{p^i}}) \text{ to order } 1 \\ & s_{(1+p^i)/2}([v_1] * y_{2(1+p^i)}) \text{ to order } 1 \\ & s_{(p^i+p^i)/2}([v_1] * y_{2(p^i+p^i)}) \text{ to order } 1 \\ & s_{(p^i+p^i)/2}([v_1] * (z_{2(p^i+p^i)} - y_{2(p^i+p^i)})) \text{ to order } \infty \\ & \text{and } s_{(p^i+1)/2, (p^i-1)/2}(z_{2(p^i+p^i)} - y_{2(p^i+p^i)}) \text{ to order } 1 \\ & \text{but } s_{(p^i+1)/2, (p^i-1)/2}([v_1] * (v_j \cdot y_{2(1+p^i)})) \text{ to order } 2 \end{aligned}$$

Theorem 2 follows from these six facts, (1) from the first three and (2) from the last three. In more detail, the last three facts imply that the image of  $z_{2(p^i+p^i)} - y_{2(p^i+p^i)}$  can be distinguished from the image of  $y_{2(p^i+p^i)}$  and from the images of degree-2( $p^i + p^i$ ) products of lower degree generators by the vanishing of the number  $s_{(p^i+p^i)/2}$  together with the nonvanishing mod  $p^2$  of the number  $s_{(p^i+1)/2, (p^i-1)/2}$ .

### 3. Cayley plane bundles

In this section we summarize work of Borel & Hirzebruch on characteristic classes of homogeneous spaces which we will use in the next section to prove Theorem 1.

The Cayley plane is the homogeneous space  $\mathbf{CaP}^2 = F_4/\text{Spin}(9)$ . Much of what follows applies to any bundle with fiber a homogeneous space  $G/H$ , though, so we begin in that generality and later specialize to the case  $G/H = F_4/\text{Spin}(9)$ .

Throughout this section let  $G$  be a compact connected Lie group, let  $i_{H,G} : H \hookrightarrow G$  be a maximal rank subgroup, and let  $i_{T,H} : T \rightarrow H$  and  $i_{T,G} : T \rightarrow G$  be the inclusions of a common maximal torus.

Every  $G/H$  bundle (with structure group  $G$ ) pulls back from the universal  $G/H$  bundle  $G/H \rightarrow BH \rightarrow BG$ . That is, every  $G/H$  bundle fits into a pullback square:

$$\begin{array}{ccc} E_f & \xrightarrow{\tilde{f}} & BH \\ \pi_f \downarrow & & \downarrow \text{Bi}_{H,G} \\ Z & \xrightarrow{f} & BG \end{array}$$

where  $f$  is unique up to homotopy and  $\tilde{f}$  is canonically determined by  $f$ .

Let  $\eta$  denote the relative tangent bundle of  $BH \rightarrow BG$ . Then the relative tangent bundle of  $E_f \rightarrow Z$  is the pullback  $\tilde{f}^*(\eta)$  and there is an exact sequence:

$$0 \rightarrow \tilde{f}^*(\eta) \rightarrow TE_f \rightarrow \pi_f^*TZ \rightarrow 0$$

This implies for instance that  $p_1(TE_f) = \pi_f^*p_1(TZ) + \tilde{f}^*p_1(\eta)$ .

The characteristic classes of  $\eta$ , or rather their pullbacks to  $H^*(BT, \mathbf{Z})$ , may be computed using the beautiful methods of [BH58] (see especially Theorem 10.7). For example, the pullbacks of the first Pontrjagin class  $p_1(\eta)$  and more generally the characteristic class  $s_I(\eta)$  are given by the formulas:

$$\text{Bi}_{T,H}^*p_1(\eta) = \sum r_i^2 \quad \text{Bi}_{T,H}^*s_I(\eta) = s_I(r_1^2, \dots, r_m^2)$$

where  $(\pm r_1, \dots, \pm r_m)$  are the roots of  $G$  complementary to those of  $H$  regarded as elements of  $H^*(BT, \mathbf{Z})$ .

Borel-Hirzebruch's Lie-theoretic description [BH58, BH59] of the pushforward:

$$\mathrm{Bi}_{H,G*} : H^*(BH, \mathbf{Z}) \rightarrow H^*(BG, \mathbf{Z})$$

is essential to proving Theorem 1. In order to state their result we need to introduce some notation.

Associated to  $G$  is a generalized Euler class  $\tilde{e}(G/T) \in H^*(BT, \mathbf{Z})$ . It makes sense to call it that because it restricts to the Euler class of the fiber  $G/T$  of the bundle  $BT \rightarrow BG$ . Up to sign  $\tilde{e}(G/T)$  is the product of a set of positive roots of  $G$ , regarded as elements of  $H^*(BT, \mathbf{Z})$ . More precisely it is the product of the roots of an invariant almost complex structure on  $G/T$ . (See [BH58, §12.3, §13.4] for more details.) Note that  $G/T$  always admits a complex structure and that although the individual roots associated to an almost complex structure depend on the almost complex structure, their product  $\tilde{e}(G/T)$  does not.

**Theorem 3** (Borel-Hirzebruch, Theorem 20.3 of [BH59]). *If  $t \in H^*(BT, \mathbf{Z})$  then:*

$$\mathrm{Bi}_{T,G}^* \mathrm{Bi}_{T,G*}(t) = \frac{1}{\tilde{e}(G/T)} \sum_{w \in W(G)} \mathrm{sgn}(w) w(t)$$

where  $W(G)$  denotes the Weyl group of  $G$ .

**Corollary 4.** *If  $h \in H^*(BH, \mathbf{Z})$  then:*

$$\mathrm{Bi}_{T,G}^* \mathrm{Bi}_{H,G*}(h) = \sum_{[w] \in W(G)/W(H)} w \left( \frac{\tilde{e}(H/T)}{\tilde{e}(G/T)} \mathrm{Bi}_{T,H}^*(h) \right)$$

where the sum runs over the cosets of  $W(H)$  in  $W(G)$ .

PROOF. Since  $\mathrm{Bi}_{T,H*} \tilde{e}(H/T) = \chi(H/T) = |W(H)| \in H^0(BH, \mathbf{Z})$ , write:

$$\mathrm{Bi}_{T,G}^* \mathrm{Bi}_{H,G*}(h) = \mathrm{Bi}_{T,G}^* \mathrm{Bi}_{H,G*} \left( \frac{\mathrm{Bi}_{T,H*}(\tilde{e}(H/T))}{|W(H)|} \cdot h \right)$$

Apply the projection formula to obtain:

$$\begin{aligned} \mathrm{Bi}_{T,G}^* \mathrm{Bi}_{H,G*}(h) &= \frac{1}{|W(H)|} \mathrm{Bi}_{T,G}^* \mathrm{Bi}_{H,G*} \mathrm{Bi}_{T,H*} (\tilde{e}(H/T) \cdot \mathrm{Bi}_{T,H}^*(h)) \\ &= \frac{1}{|W(H)|} \mathrm{Bi}_{T,G}^* \mathrm{Bi}_{T,G*} (\tilde{e}(H/T) \cdot \mathrm{Bi}_{T,H}^*(h)) \end{aligned}$$

Apply Theorem 3 to obtain:

$$\mathrm{Bi}_{T,G}^* \mathrm{Bi}_{H,G*}(h) = \frac{1}{|W(H)|} \cdot \frac{1}{\tilde{e}(G/T)} \sum_{w \in W(G)} \mathrm{sgn}(w) w(\tilde{e}(H/T) \cdot \mathrm{Bi}_{T,H}^*(h))$$

Since  $w(\tilde{e}(G/T)) = \mathrm{sgn}(w) \tilde{e}(G/T)$ :

$$\mathrm{Bi}_{T,G}^* \mathrm{Bi}_{H,G*}(h) = \frac{1}{|W(H)|} \sum_{w \in W(G)} w \left( \frac{\tilde{e}(H/T)}{\tilde{e}(G/T)} \mathrm{Bi}_{T,H}^*(h) \right)$$

Since  $W(G)$  acts on  $H^*(BT, \mathbf{Z})$  by ring homomorphisms, since if  $w \in W(H)$  then  $w(\tilde{e}(H/T)) = \mathrm{sgn}(w) \tilde{e}(H/T)$  and  $w(\tilde{e}(G/T)) = \mathrm{sgn}(w) \tilde{e}(G/T)$ , and since  $\mathrm{Bi}_{T,H}^*$  maps to the  $W(H)$ -invariant subring of  $H^*(BT, \mathbf{Z})$ , this sum can be written over the cosets of  $W(H)$  in  $W(G)$ :

$$\mathrm{Bi}_{T,G}^* \mathrm{Bi}_{H,G*}(h) = \sum_{[w] \in W(G)/W(H)} w \left( \frac{\tilde{e}(H/T)}{\tilde{e}(G/T)} \mathrm{Bi}_{T,H}^*(h) \right) \quad \square$$

Now we specialize to Cayley plane bundles. Let  $F_4$  denote the 1-connected compact Lie group of type  $F_4$ . The extended Dynkin diagram of  $F_4$  is:

$$\bullet \text{ --- } \circ \text{ --- } \circ \Rightarrow \circ \text{ --- } \circ$$

$$\begin{array}{ccccc} & & a_1 & a_2 & a_3 & a_4 \\ -\tilde{a} & & & & & \end{array}$$

The corresponding simple roots can be taken to be:

$$a_1 = e_2 - e_3 \quad a_2 = e_3 - e_4 \quad a_3 = e_4 \quad a_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$$

Since the coefficient of  $a_4$  in the maximal root  $\tilde{a} = 2a_1 + 3a_2 + 4a_3 + 2a_4 = e_1 + e_2$  is prime, a theorem of Borel & de Siebenthal [BDS49] implies that erasing  $a_4$  from the extended Dynkin diagram gives the Dynkin diagram of a subgroup:

$$\circ \text{ --- } \circ \text{ --- } \circ \Rightarrow \circ$$

$$\begin{array}{cccc} -\tilde{a} & a_1 & a_2 & a_3 \end{array}$$

Since  $F_4$  is 1-connected this subgroup is  $\text{Spin}(9)$ , the 1-connected double cover of  $\text{SO}(9)$ . The Cayley plane is the homogeneous space  $\mathbf{CaP}^2 = F_4/\text{Spin}(9)$ .

In terms of the standard basis  $e_1, \dots, e_4$ , the roots of  $\text{Spin}(9)$  are:

$$\begin{cases} \pm e_i & 1 \leq i \leq 4 \\ \pm e_i \pm e_j & 1 \leq i < j \leq 4 \end{cases}$$

The roots of  $F_4$  are those of  $\text{Spin}(9)$  together with the complementary roots:

$$\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$$

The following positive roots define an almost complex structure on  $\text{Spin}(9)/T$ :

$$\begin{cases} e_i & 1 \leq i \leq 4 \\ e_i \pm e_j & 1 \leq i < j \leq 4 \end{cases}$$

These positive roots together with the following complementary positive roots define an almost complex structure on  $F_4/T$ :

$$r_i := \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4) \quad \text{for } 1 \leq i \leq 8$$

In order to identify these roots with elements of  $H^2(BT, \mathbf{Z}) \cong \text{Hom}(\Gamma, \mathbf{Z})$  note that in general a Lie group's lattice of integral forms is sandwiched somewhere between its root and weight lattices:

$$R \subset \text{Hom}(\Gamma, \mathbf{Z}) \subset W \subset LT^*$$

But in the case of  $F_4$  all three lattices coincide (because the Cartan matrix of  $F_4$  has determinant 1).

Finally note that if  $s_i$  denotes reflection across the hyperplane orthogonal to the simple root  $a_i$  then the 3 cosets of  $W(\text{Spin}(9))$  in  $W(F_4)$  can be represented by the reflections  $\{1, s_4, s_4 s_3 s_4\}$  which act on  $e_1, \dots, e_4$  according to the matrices:

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \right\}$$

In particular these reflections act on the set of positive complementary roots  $r_i$  by:

$$\begin{aligned} \{r_i\} &= \{\tfrac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\} \\ s_4(\{r_i\}) &= \{e_1, e_2, e_3, e_4, \tfrac{1}{2}(e_1 + e_2 + e_3 - e_4), \tfrac{1}{2}(e_1 + e_2 - e_3 + e_4), \\ &\quad \tfrac{1}{2}(e_1 - e_2 + e_3 + e_4), \tfrac{1}{2}(-e_1 + e_2 + e_3 + e_4)\} \\ s_4 s_3 s_4(\{r_i\}) &= \{e_1, e_2, e_3, e_4, \tfrac{1}{2}(e_1 + e_2 + e_3 + e_4), \tfrac{1}{2}(e_1 + e_2 - e_3 - e_4), \\ &\quad \tfrac{1}{2}(e_1 - e_2 + e_3 - e_4), \tfrac{1}{2}(-e_1 + e_2 + e_3 - e_4)\} \end{aligned}$$

**Corollary 5.**

$$\text{Bi}_{T, F_4}^* \text{Bi}_{\text{Spin}(9), F_4} s_I(\eta) = \frac{s_I(r_1^2, \dots, r_8^2)}{\prod_i r_i} + s_4 \left( \frac{s_I(r_1^2, \dots, r_8^2)}{\prod_i r_i} \right) + s_4 s_3 s_4 \left( \frac{s_I(r_1^2, \dots, r_8^2)}{\prod_i r_i} \right)$$

where the complementary roots  $r_i = \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)$  are regarded as elements of  $H^2(\text{BT}, \mathbf{Z})$  and  $s_4, s_4 s_3 s_4$  act on them as described above.

#### 4. Proof of Theorem 1

The purpose of this section is to prove the following theorem, which was already stated in the introduction.

**Theorem 1.** *Cayley plane bundles generate the kernel of the Witten genus tensor  $\mathbf{Z}[\frac{1}{6}]$ :*

$$\phi_W : \pi_* \text{MO}\langle 8 \rangle[\tfrac{1}{6}] \rightarrow \pi_* \text{tmf}[\tfrac{1}{6}] \cong \mathbf{Z}[\tfrac{1}{6}][\mathbf{G}_4, \mathbf{G}_6]$$

We shall do this by showing that the set  $S$  of (bordism classes of) total spaces  $E$  of bundles  $\mathbf{CaP}^2 \rightarrow E \rightarrow W$  with connected structure group, where  $E$  is string but  $W$  is not necessarily string, satisfy the conditions of Theorem 2 except in dimensions 8 and 12. Theorem 1 will then follow by the following theorem.

**Theorem.** *If  $\mathbf{CaP}^2 \rightarrow E \rightarrow W$  is a Cayley plane bundle with connected structure group then the Witten genus of  $E$  vanishes.*

This result was often proved in the 1990's—by Jung, Kreck-Singhof-Stolz, Dessai, Höhn—but rarely published. Rainer Jung's proof, which has yet to appear in print, used the work of Borel-Hirzebruch summarized above to show that the vanishing of the Witten genus on Cayley plane bundles is equivalent to the Jacobi triple identity for the Weierstrass sigma function. A little later Anand Dessai proved, using results of Kefeng Liu [Liu92], that if  $S^3$  acts nontrivially on a string manifold  $E$  then the Witten genus of  $E$  vanishes. (This generalizes the theorem above since  $S^3$  acts nontrivially on the total space of any Cayley plane bundle.) Dessai's work appeared in the preprint [Des94], in his PhD thesis [Des96], and in the conference proceedings [Des09]. Around the same time Gerald Höhn proved, again using results of Liu, that the Witten genus of any string homogeneous manifold vanishes. These results helped inspire Stephan Stolz's conjecture [Sto96] (see Theorem 3.1) that the Witten genus of a closed  $4k$ -dimensional string manifold vanishes iff it admits a Riemannian metric of positive Ricci curvature. Jung and Dessai in fact proved the rational version of Theorem 1:

**Theorem.** *Cayley plane bundles generate the kernel of the Witten genus tensor  $\mathbf{Q}$ :*

$$\phi_W : \pi_* \text{MO}\langle 8 \rangle \otimes \mathbf{Q} \rightarrow \pi_* \text{tmf} \otimes \mathbf{Q} \cong \mathbf{Q}[\mathbf{G}_4, \mathbf{G}_6]$$

(The author thanks Dessai for informing him of the history of these results.)

Let  $i : V^m(d_1, \dots, d_r) \hookrightarrow \mathbf{CP}^{m+r}$  denote a smooth complete intersection of degree  $(d_1, \dots, d_r)$  and complex dimension  $m$ . Consider the  $\mathbf{CaP}^2$  bundle pulling back from the universal bundle  $\mathbf{CaP}^2 \rightarrow \mathrm{BSpin}(9) \rightarrow \mathrm{BF}_4$  by a classifying map  $g$  of the form:

$$n + (a + b) = a \cdot 2^2 + b \cdot 3^2$$

PROOF. This follows by induction since:

$$14 + 3 = 2^2 + 2^2 + 3^2 \quad 15 + 5 = 2^2 + 2^2 + 2^2 + 2^2 + 2^2 \quad 16 + 2 = 3^2 + 3^2$$

and since:

$$n + (a + b) = a \cdot 2^2 + b \cdot 3^2 \quad \implies \quad (n + 3) + (a + 1 + b) = (a + 1) \cdot 2^2 + b \cdot 3^2 \quad \square$$

As an aside, the values for  $a$  and  $b$  constructed in the proof are:

$$a(n) = 3n - 8 \lceil n/3 \rceil \quad b(n) = 3 \lceil n/3 \rceil - n$$

Although the preceding lemma suffices to prove the results of this paper, the reader may find the reliance on complete intersections of arbitrarily high codimension unsatisfying. It is therefore worth noting that the following replacement for Lemma 6 would make it possible to prove the results of this paper using complete intersections of codimension  $\leq 4$ .

**Conjecture 7.** *If  $n \geq 25$  then the GCD:*

$$\text{GCD} \left\{ \prod_{i=1}^4 d_i \mid 4n + 4 + 1 = \sum_{i=1}^4 d_i^2, d_i > 0 \right\}$$

*has the form  $2^a 3^b$  with  $a + b > 0$ . In fact as  $n$  increases from 25, this GCD takes the values:*

$$2^4 \cdot 3 \quad 2^3 \quad 2^4 \cdot 3^2 \quad 2^3 \cdot 3 \quad 2^4 \quad 2^3 \cdot 3^2$$

*and then repeats from the beginning.*

We have to carefully choose the degrees  $(d_1, \dots, d_r)$  and  $(d'_1, \dots, d'_{r'})$  to ensure that the total space  $E$  admits a string structure. However, these degrees have little effect on the Pontrjagin number  $s_n[E]$  which we compute next. Indeed, for dimension reasons:

$$s_n[E] = (i \times i')^* f^* \text{BiSpin}(9)_{F_4*} s_n(\eta)$$

Since the base space  $W$  is a product of complete intersections, the pullback  $(i \times i')^* x_1^m x_2^{m'}$  equals  $(\prod_j d_j)(\prod_{j'} d'_{j'})$  times the fundamental class  $[W]$ . So the key is to compute the coefficients of the polynomial  $f^* \text{BiSpin}(9)_{F_4*} s_n(\eta)$  or, rather, their GCD as a function of  $n$ . This calculation lies at the heart of this paper. (It was the smoking gun which led to Theorem 1.)

**Proposition 8.**

$$f^* \text{BiSpin}(9)_{F_4*} s_n(\eta) = 2n_f^{2n-8} \sum_{k=2}^{n-2} \left[ \binom{2n}{2} - \binom{2n}{2k} \right] x_1^{2k-4} x_2^{2n-2k-4}$$

PROOF. Since the polynomial in question is homogeneous in  $n_f x_1$  and  $n_f x_2$ , we can, without loss of generality, simplify notation by setting  $n_f = 1$  and  $(x_1, x_2) = (x, 1)$ .

Corollary 5 gives the polynomial in the form of a power series:

$$\begin{aligned} & -\frac{1}{x^4} (1 + x^2 + x^4 + \dots) \\ & \cdot \left[ \underbrace{-2 + (x+1)^{2n} + (x-1)^{2n}}_{-2 + (x+1)^{2n} + (x-1)^{2n} + 2 \binom{2n}{2}} - x^2 \left[ \overbrace{-2 + (x+1)^{2n} + (x-1)^{2n} + 2 \binom{2n}{2}} \right] \right. \\ & \quad \left. + x^{2n} \left[ 2 \binom{2n}{2} - 2 \right] + 2x^{2n+2} \right] \end{aligned}$$

The bracketed quantities differ by  $2\binom{2n}{2}$  so the power series simplifies to the polynomial:

$$-\frac{1}{x^4} \cdot \left[ -2 + (x+1)^{2n} + (x-1)^{2n} - 2\binom{2n}{2}(x^2 + x^4 + \dots + x^{2n-2}) - 2x^{2n} \right]$$

which simplifies further to:

$$2 \sum_{k=2}^{n-1} \left[ \binom{2n}{2} - \binom{2n}{2k} \right] x^{2k-4} \quad \square$$

**Proposition 9.** For any integer  $n \geq 4$  and any odd prime  $p$ :

$$\text{ord}_p \left[ \text{GCD}_{1 \leq k \leq n-1} \left\{ \binom{2n}{2} - \binom{2n}{2k} \right\} \right] = \begin{cases} 1 & \text{if } 2n = p^i - 1 \text{ or } 2n = p^i + p^j \text{ for some } 0 \leq i \leq j \\ 0 & \text{otherwise} \end{cases}$$

The key behind this is the following lemma.

**Lemma 10.** For any integer  $n > 1$  and any odd prime  $p$ :

$$\text{ord}_p \left[ \text{GCD}_{0 < k < n} \binom{2n}{2k} \right] = \begin{cases} 1 & \text{if } 2n = p^i + p^j \text{ for some } 0 \leq i \leq j \\ 0 & \text{otherwise} \end{cases}$$

It is worth comparing this result to the better known result that for any integer  $n > 1$  and any prime  $p$ :

$$\text{ord}_p \left[ \text{GCD}_{0 < k < n} \binom{n}{k} \right] = \begin{cases} 1 & \text{if } n = p^i \text{ for some integer } i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Notice that, for any given integer  $n > 1$ , at most one prime divides the latter GCD whereas several primes may divide the former. For example, if  $n = 7$  then  $2n = 7^1 + 7^1 = 13^0 + 13^1$  and indeed  $\text{GCD}_{0 < k < 7} \binom{14}{2k} = 7 \cdot 13$ .

**PROOF OF LEMMA 10.** By Kummer's theorem  $\binom{2n}{2k}$  is divisible by  $p$  if and only if there is at least 1 carry when adding  $2k$  to  $2n - 2k$ . Consider the base- $p$  expansion  $\sum n_i p^i$  of an even integer  $2n$ . If there is a digit  $n_i \geq 2$  then there is no carry when adding  $2p^i$  to  $2n - 2p^i$ . If there are 2 distinct nonzero digits  $n_i, n_j$  then there is no carry when adding  $p^i + p^j$  to  $2n - p^i - p^j$ . If  $2n = p^i + p^j$  and  $0 < 2k < 2n$  then there is always a carry when adding  $2k$  to  $2n - 2k$ , even if  $i = j$ . These 3 facts together imply the first part of the lemma. The second part of the lemma follows from the fact that if  $j > 0$  then there is precisely 1 carry when adding  $(p-1)p^{j-1}$  to  $p^i + p^j - (p-1)p^{j-1}$ . (If  $j = 0$  then the second part of the lemma is vacuous.)  $\square$

**PROOF OF PROPOSITION 9.** If an odd prime  $p$  divides the GCD then all the binomial coefficients  $\binom{2n}{2k}$  for  $0 < 2k < 2n$  must be congruent mod  $p$ . If they are all congruent to 0 mod  $p$  then Lemma 10 applies and  $2n = p^i + p^j$  for some  $0 \leq i \leq j$ . So suppose that the binomial coefficients are all nonzero mod  $p$ . By Kummer's theorem this happens precisely when for each  $0 < 2k < 2n$  there are no carries when adding  $2k$  to  $2n - 2k$ . This in turn happens precisely when  $2n = l \cdot p^i - 1$  for some  $i > 0$  and some (odd)  $0 < l < p$ . According to Lucas's theorem (see [Gra97, §1]), if  $l > 1$  then:

$$\binom{l \cdot p^i - 1}{p^i + 1} \equiv \binom{p-1}{1} \binom{p-1}{0} \cdots \binom{p-1}{0} \binom{l-1}{1} \equiv 1 - l \pmod{p}$$

However:

$$\binom{l \cdot p^i - 1}{2} \equiv 1 \pmod{p}$$

So all the binomial coefficients can be congruent mod  $p$  only if  $l = 1$ , and indeed the congruence  $(1+x)^{p^i} \equiv 1+x^{p^i} \pmod{p}$  implies that:

$$(1+x)^{p^{i-1}} \equiv (1+x^{p^i})(1+x)^{-1} = 1-x+x^2-x^3+\dots+x^{p^i-1} \pmod{p}$$

and hence that:

$$\binom{p^i-1}{2k} \equiv 1 \pmod{p}$$

for all  $0 < 2k < p^i - 1$ .

It remains to show that the GCD is never divisible by  $p^2$  for  $p$  odd. By the preceding argument it remains only to show this when  $2n = p^i + p^j$  or  $2n = p^i - 1$  for  $0 \leq i \leq j$ . Remember that by assumption  $2n \geq 16$ .

Suppose first that  $2n = p^i + p^j$ . If  $i > 1$  then there are at least 2 carries when adding 2 to  $p^i + p^j - 2$ ; so by Kummer's theorem  $\binom{2n}{2}$  is congruent to 0 mod  $p^2$  while by Lemma 10  $\binom{2n}{2k}$  is nonzero mod  $p^2$  for some  $0 < 2k < 2n$ . If  $i \leq 1$  then there are 4 possible values of  $p^i + p^j$  which can possibly be  $\geq 16$ , namely  $p+1, p+p, p^2+1, p^2+p$ . The 3rd value can be handled as when  $i > 1$ . The 1st, 2nd and 4th values can be handled using the following elementary congruences mod  $p^2$ :

$$\binom{p+1}{2} - \binom{p+1}{4} \equiv \frac{5}{12}p \quad \binom{2p}{2} - \binom{2p}{4} \equiv -\frac{1}{2}p \quad \binom{p^2+p}{2} - \binom{p^2+p}{4} \equiv -\frac{1}{4}p$$

The coefficient  $\frac{5}{12}$  is not a problem since  $2n = p+1 \geq 16$  only if  $p \geq 17$ .

Suppose now that  $2n = p^i - 1$ . Consider the following congruences mod  $p^2$ :

$$\binom{p^i-1}{2} \equiv 1 - \frac{3}{2}p^i \quad \binom{p-1}{4} \equiv 1 - \frac{25}{12}p \quad \binom{p^i-1}{p^{i-1}+p^{i-2}} \equiv 1 - p$$

The 1st and 2nd are immediate, and subtracting them gives the desired result for  $i = 1$ . (The resulting coefficient  $-\frac{3}{2} + \frac{25}{12} = \frac{7}{12}$  of  $p$  is not a problem since  $2n = p-1 \geq 16$  only if  $p \geq 17$ .) Subtracting the 3rd congruence from the 1st gives the desired result when  $i \geq 2$  but proving the 3rd congruence is more subtle. Here, and quite often in what follows, we rely on the following powerful theorem.

**Granville's theorem ([Gra97, Theorem 1]).** Suppose that a prime power  $p^q$  and positive integers  $n = m + r$  are given. Write  $n = n_0 + n_1p + \dots + n_dp^d$  in base  $p$ , and let  $N_j$  be the least positive residue of  $[n/p^j] \pmod{p^q}$  for each  $j \geq 0$  (so that  $N_j = n_j + n_{j+1}p + \dots + n_{j+q-1}p^{q-1}$ ); also make the corresponding definitions for  $m_j, M_j, r_j, R_j$ . Let  $e_j$  be the number of indices  $i \geq j$  for which  $n_i < m_i$  (that is, the number of 'carries', when adding  $m$  and  $r$  in base  $p$ , on or beyond the  $j$ th digit). Then:

$$\frac{1}{p^{e_0}} \equiv (\pm 1)^{e_{q-1}} \left( \frac{(N_0!)_p}{(M_0!)_p(R_0!)_p} \right) \left( \frac{(N_1!)_p}{(M_1!)_p(R_1!)_p} \right) \dots \left( \frac{(N_d!)_p}{(M_d!)_p(R_d!)_p} \right) \pmod{p^q}$$

where  $(\pm 1)$  is  $(-1)$  except if  $p = 2$  and  $q \geq 3$ . Here  $(n!)_p$  denotes the product of those integers  $\leq n$  which are not divisible by  $p$ .



We need to show that the 3rd congruence holds for  $i \geq 2$  but assume first that  $i \geq 3$ . Then according to Granville's theorem the binomial coefficient  $\binom{p^i-1}{p^{i-1}+p^{i-2}}$  is congruent to:

$$\frac{((p^2-1)!)_p}{(p!)_p \cdot ((p^2-p-1)!)_p} \cdot \frac{((p^2-1)!)_p}{((p+1)!)_p \cdot ((p^2-p-2)!)_p} \cdot \frac{((p-1)!)_p}{(1!)_p \cdot ((p-2)!)_p} \mod p^2$$

Gathering common factors gives:

$$\begin{aligned} \binom{p^i-1}{p^{i-1}+p^{i-2}} &\equiv \left( \frac{(1-p)(2-p) \cdots ((p-1)-p)}{(p!)_p} \right)^2 \cdot \frac{p^2-p-1}{p+1} \cdot (p-1) \mod p^2 \\ &\equiv \left( 1 - \underbrace{p\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1}\right)}_{\equiv 0} \right)^2 \cdot (1-p) \mod p^2 \end{aligned}$$

The bracketed quantity is congruent to 0 mod  $p^2$  since by Wolstenholme's theorem [HW79, Theorem 116] ("Wolstenholme... he was despondent and dissatisfied and consoled himself with mathematics and opium"—Sir Leslie Stephen, Virginia Woolf's father) the  $(p-1)$ st harmonic number is congruent to 0 mod  $p^2$  for  $p > 3$  and to  $2p$  for  $p = 3$ . Thus we obtain:

$$\binom{p^i-1}{p^{i-1}+p^{i-2}} \equiv 1^2 \cdot (1-p) = 1-p \mod p^2$$

If  $i = 2$  then the first factor in the congruence provided by Granville's theorem disappears, and the square in the following congruences therefore does too but, since  $1^2 = 1$ , this does not affect the final result.  $\square$

**Construction of the elements  $N^{2(p^i+p^j)}$ .** Throughout this section let  $p > 3$  and  $0 < i < j$  be arbitrary but fixed. To simplify notation let:

$$(n_1, n_2) = \left( \frac{1}{2}(p^j-1), \frac{1}{2}(p^i+1) \right)$$

Our goal is to construct an element  $N^{4(n_1+n_2)}$  with:

$$\begin{aligned} s_{n_1+n_2}[N^{4(n_1+n_2)}] &= 0 \\ s_{n_1, n_2}[N^{4(n_1+n_2)}] &\not\equiv 0 \mod p^2 \end{aligned}$$

To do this, we will construct two  $\mathbf{CaP}^2$  bundles  $E_1$  and  $E_2$  and define:

$$N^{4(n_1+n_2)} = \text{LCM}(s_{n_1+n_2}[E_1], s_{n_1+n_2}[E_2]) \cdot \left( \frac{E_1}{s_{n_1+n_2}[E_1]} - \frac{E_2}{s_{n_1+n_2}[E_2]} \right)$$

Then  $s_{n_1+n_2}[N^{4(n_1+n_2)}] = 0$ , so all that will remain will be to show that  $s_{n_1, n_2}[N^{4(n_1+n_2)}] \not\equiv 0 \mod p^2$ . To do so, it will suffice to show that:

$$\begin{aligned} s_{n_1, n_2}[E_1] &\equiv 0 \mod p^2 \\ s_{n_1, n_2}[E_2] &\not\equiv 0 \mod p^2 \\ \text{ord}_p s_{n_1+n_2}[E_1] &\leq \text{ord}_p s_{n_1+n_2}[E_2] \end{aligned}$$

Above we saw that the characteristic number  $s_n[E]$  depends only on the image of  $s_n(\eta)$  in  $H^*(E)$  and *not* on the Pontrjagin classes of the base  $W$ . The characteristic number  $s_{n_1, n_2}[E]$

is more subtle, however. Indeed, for a bundle  $\mathbf{CaP}^2 \rightarrow E \xrightarrow{\pi} W$  classified as before by a map  $g : W \rightarrow \mathbf{BF}_4$ , we have:

$$\begin{aligned} s_{n_1, n_2}(\mathbf{TE}) &= \tilde{g}^* s_{n_1, n_2}(\eta) \\ &\quad + \pi^* s_{n_1}(\mathbf{TW}) \cdot \tilde{g}^* s_{n_2}(\eta) \\ &\quad + \pi^* s_{n_2}(\mathbf{TW}) \cdot \tilde{g}^* s_{n_1}(\eta) \\ &\quad + \pi^* s_{n_1, n_2}(\mathbf{TW}) \end{aligned}$$

Applying the  $H^*(W)$ -module homomorphism  $B\pi_*$  (which decreases degrees by 16) gives:

$$\begin{aligned} B\pi_* s_{n_1, n_2}(\mathbf{TE}) &= g^* \mathbf{Bi}_* s_{n_1, n_2}(\eta) \\ &\quad + s_{n_1}(\mathbf{TW}) \cdot g^* \mathbf{Bi}_* s_{n_2}(\eta) \\ &\quad + s_{n_2}(\mathbf{TW}) \cdot g^* \mathbf{Bi}_* s_{n_1}(\eta) \end{aligned}$$

To compute the last two terms, note that the 2nd exact sequence of vector bundles on p. 15 implies that:

$$\begin{aligned} s_n(\mathbf{TW}) &= s_n(\mathbf{TV}^m(d_1, \dots, d_r) \times \mathbf{TV}^{m'}(d'_1, \dots, d'_{r'})) \\ &= i^* \left( s_n(\mathbf{CP}^{m+r}) - \sum_j s_n \mathbf{O}(d_j) \right) + i'^* \left( s_n(\mathbf{CP}^{m'+r'}) - \sum_{j'} s_n \mathbf{O}(d'_{j'}) \right) \\ &= (i \times i')^* \left[ \left( m + r + 1 - \sum_j d_j^{2n} \right) x_1^{2n} + \left( m' + r' + 1 - \sum_{j'} (d'_{j'})^{2n} \right) x_2^{2n} \right] \end{aligned}$$

Let  $E_1$  be the  $\mathbf{CaP}^2$  bundle obtained by taking:

$$(m, m') = (2n_1 - 2, 2n_2 - 6) = (p^j - 3, p^i - 5)$$

in the construction of  $E$  above. Then for dimension reasons:

$$s_{n_1}(\mathbf{TV}^m) = s_{n_1}(\mathbf{TV}^{m'}) = s_{n_2}(\mathbf{TV}^{m'}) = 0$$

and by Proposition 8:

$$\begin{aligned} B\pi_* s_{n_1, n_2}(\mathbf{TE}_1) &= \\ &g^* \mathbf{Bi}_* s_{n_1, n_2}(\eta) + \left( m + r + 1 - \sum_j d_j^{2n_2} \right) \cdot \left[ \binom{p^j - 1}{2} - \binom{p^j - 1}{p^j - p^i} \right] \cdot (i \times i')^* x_1^m x_2^{m'} \end{aligned}$$

Part (1) of Corollary 13 below shows that  $g^* \mathbf{Bi}_* s_{n_1, n_2}(\eta) \equiv 0 \pmod{p^2}$  and Granville's theorem can be used to show that both binomial coefficients are congruent to 1 mod  $p^2$  so:

$$B\pi_* s_{n_1, n_2}[\mathbf{E}_1] \equiv 0 \pmod{p^2}$$

Let  $E_2$  be the  $\mathbf{CaP}^2$  bundle obtained by taking:

$$(m, m') = (p^{j-1} - 3, p^j - p^{j-1} + p^i - 5)$$

in the construction of  $E$  above. Then for dimension reasons  $s_{n_1}(TV^m) = s_{n_1}(TV^{m'}) = 0$ . If  $i = j - 1$  then  $s_{n_2}(TV^m) = 0$  as well. So by Proposition 8:

$$\begin{aligned} B\pi_* s_{n_1, n_2}(TE_2) = & g^* \text{Bi}_* s_{n_1, n_2}(\eta) \\ & + \left( m + r + 1 - \sum_j d_j^{p^j+1} \right) \cdot \left[ \binom{p^j-1}{2} - \binom{p^j-1}{p^{j-1}-p^i} \right] \cdot (1 - \delta_{i=j-1}) \\ & + \left( m' + r' + 1 - \sum_{j'} (d'_{j'})^{p^{j'}+1} \right) \cdot \left[ \binom{p^j-1}{2} - \binom{p^j-1}{p^{j-1}+1} \right] \cdot (i \times i')^* x_1^m x_2^{m'} \end{aligned}$$

(Here  $\delta_P$  equals 1 if  $P$  is true and equals 0 otherwise.) Granville's theorem can be used to show that the first three binomial coefficients are congruent to 1 mod  $p^2$  while the last is congruent to  $1 - p$  mod  $p^2$  so:

$$B\pi_* s_{n_1, n_2}(TE_2) \equiv g^* \text{Bi}_* s_{n_1, n_2}(\eta) + \left( m' + r' + 1 - \sum_{j'} (d'_{j'})^{p^{j'}+1} \right) \cdot p \cdot (i \times i')^* x_1^m x_2^{m'} \pmod{p^2}$$

By Fermat's little theorem:

$$\left( m' + r' + 1 - \sum_{j'} (d'_{j'})^{p^{j'}+1} \right) \equiv \left( m' + r' + 1 - \sum_{j'} (d'_{j'})^2 \right) \pmod{p}$$

Recall that the degrees  $(d'_1, \dots, d'_{r'})$  are chosen (say using Lemma 6) to make the latter quantity equal  $-4n_f$  (since this makes  $p_1(TE_2) = 0$ ). So the particular degrees chosen are irrelevant here and:

$$B\pi_* s_{n_1, n_2}(TE_2) \equiv g^* \text{Bi}_* s_{n_1, n_2}(\eta) - 4n_f \cdot p \cdot (i \times i')^* x_1^m x_2^{m'} \pmod{p^2}$$

By Part (2) of Corollary 13 below  $g^* \text{Bi}_* s_{n_1, n_2}(\eta) \equiv 8p \cdot n_f^{m+m'} \cdot (i \times i')^* x_1^m x_2^{m'} \pmod{p^2}$  so:

$$B\pi_* s_{n_1, n_2}(TE_2) \equiv (8n_f^{p^i+p^j-8} - 4n_f) \cdot p \cdot (i \times i')^* x_1^m x_2^{m'} \pmod{p^2}$$

By Fermat's little theorem:

$$B\pi_* s_{n_1, n_2}(TE_2) \equiv 4n_f(2n_f^{-7} - 1) \cdot p \cdot (i \times i')^* x_1^m x_2^{m'} \pmod{p^2}$$

Since  $W$  is a product of complete intersections,  $(i \times i')^* x_1^m x_2^{m'}$  equals  $(\prod_j d_j)(\prod_{j'} d'_{j'})$  times the fundamental class  $[W]$ , and the degrees are all chosen to be nonzero mod  $p$ . Determining the roots of the polynomial  $n_f^7 - 2$  mod  $p$  is a delicate task, but certainly if  $n_f \equiv 1$  mod  $p$  then:

$$B\pi_* s_{n_1, n_2}[E_2] \not\equiv 0 \pmod{p^2}$$

**Lemma 11.**

$$\text{ord}_p s_{n_1+n_2}[E_1] \leq \text{ord}_p s_{n_1+n_2}[E_2]$$

PROOF. Assuming as we did above that  $n_f \equiv 1$  mod  $p$ , it suffices by Proposition 8 to show that:

$$\text{ord}_p \left[ \binom{p^i+p^j}{2} - \binom{p^i+p^j}{p^{j-1}+1} \right] \leq \text{ord}_p \left[ \binom{p^i+p^j}{2} - \binom{p^i+p^j}{p^j+3} \right]$$

By Kummer's Theorem:

$$\text{ord}_p \binom{p^i + p^j}{2} = i \quad \text{ord}_p \binom{p^i + p^j}{p^{j-1} + 1} = i + 1 \quad \text{ord}_p \binom{p^i + p^j}{p^j + 3} = i$$

So the difference of the 1st and 2nd binomial coefficients has order  $i$  while the difference of the 1st and 3rd binomial coefficients has order  $\geq i$  (in fact it has order  $i + 2$ , as can be shown using Granville's theorem).  $\square$

The method used to prove Proposition 8 can be used to establish the following formula (which holds for any integers  $n_1 > n_2$ , not just the integers we are concerned with here).

**Proposition 12.**

$$\begin{aligned} f^* \text{Bi}_{*S_{n_1, n_2}}(\eta) = & -4n_f^{n_1 + n_2 - 8} \sum_{k=2}^{n_1 + n_2 - 1} \left[ \binom{2n_1}{2k} + \binom{2n_2}{2k} + \binom{2n_2}{2k - 2n_1} + \binom{2n_1}{2k - 2n_2} \right. \\ & + \frac{1}{2} \sum_{l=0}^k (-1)^l \binom{2n_2}{l} \binom{2n_1 - 2n_2}{2k - 2l} \\ & - \binom{2n_1}{2} \sum_{l=1}^{n_1 - 1} \binom{2n_2}{2k - 2l} - \binom{2n_2}{2} \sum_{l=1}^{n_2 - 1} \binom{2n_1}{2k - 2l} \\ & - \binom{2n_2}{2} (1 - \delta_{n_2 \leq k \leq n_1}) - \binom{2n_1}{2} (1 + \delta_{n_2 + 1 \leq k \leq n_1 - 1}) \\ & \left. + \frac{1}{2} \binom{2n_1 + 2n_2}{2} - 3\delta_{k \in \{n_1, n_2\}} \right] x_1^{2k-4} x_2^{2n_1 + 2n_2 - 2k - 4} \end{aligned}$$

where  $\delta_P$  equals 1 if  $P$  is true and equals 0 otherwise.

**Corollary 13.**

- (1) If  $(m, m') = (2n_1 - 2, 2n_2 - 6) = (p^j - 3, p^i - 5)$  then the coefficient of  $x_1^m x_2^{m'}$  in  $f^* \text{Bi}_{*S_{n_1, n_2}}(\eta)$  is congruent to 0 mod  $p^2$ .
- (2) If  $(m, m') = (p^{j-1} - 3, p^j - p^{j-1} + p^i - 5)$  then the coefficient of  $x_1^m x_2^{m'}$  in  $f^* \text{Bi}_{*S_{n_1, n_2}}(\eta)$  is congruent to  $8p \cdot n_f^{m+m'} \pmod{p^2}$ .

PROOF OF PART (1) OF COROLLARY 13. If  $(m, m') = (2n_1 - 2, 2n_2 - 6)$  then the coefficient of  $x_1^m x_2^{m'}$  is the  $k = n_1 + 1$  summand in Proposition 12. It is not difficult to show that this summand is congruent mod  $p^2$  to:

$$\begin{aligned} & 4n_f^{(p^j + p^i)/2 - 8} \left[ 0 + 0 + \frac{1}{2} p^i + 1 \right. \\ & \quad + A \\ & \quad - (2^{p^i} - 1 - \frac{1}{2} p^i) + \frac{1}{4} p^i \\ & \quad - \frac{1}{2} p^i - 1 \\ & \quad \left. - \frac{1}{4} p^i - 0 \right] \end{aligned}$$

where:

$$A = \frac{1}{2} \sum_{l=0}^{(p^j + 1)/2} (-1)^l \binom{p^i + 1}{l} \binom{p^j - p^i - 2}{p^j - 2l + 1}$$

Due to tidy pairwise cancellations, *all that remains is to show that*  $A \equiv 2^p - 1 - \frac{1}{2}p^i \pmod{p^2}$ . (Note that  $n^{p^2-p} \equiv 1 \pmod{p^2}$  for any integer  $n \not\equiv 0 \pmod{p}$  since the multiplicative group  $(\mathbb{Z}/p^2)^\times$  has order  $p^2 - p$ ; it follows by induction that  $n^{p^i} \equiv n^p \pmod{p^2}$  for any  $i > 0$ .)

(a) If  $i > 1$  then Granville's theorem can be used to show that:

$$A \equiv \sum_{r=0}^{(p-1)/2} (-1)^r \binom{p}{r} \pmod{p^2}$$

(The key is that:

$$\binom{p^i + 1}{l} \equiv \begin{cases} \binom{p+1}{l} & \text{if } i = 1 \\ \binom{p}{r} & \text{if } i > 1 \text{ and } l = rp^{i-1} \text{ or} \\ & l = rp^{i-1} + 1 \text{ with } 0 \leq r \leq p \\ 0 & \text{otherwise} \end{cases}$$

$\pmod{p^2}$ .)

By the identity  $\sum_{j=0}^k (-1)^j \binom{n}{j} = (-1)^k \binom{n-1}{k}$  (proved inductively using Pascal's rule):

$$A \equiv (-1)^{(p-1)/2} \binom{p-1}{(p-1)/2} \pmod{p^2}$$

By the eponymous congruence of Morley's ingenious 1895 paper [Mor95]:

$$A \equiv 2^{2(p-1)} \pmod{p^2}$$

The final step is to show that  $2^{2(p-1)} \equiv 2^p - 1 \pmod{p^2}$ . Write:

$$2^{2(p-1)} = (2^{p-1} + 1)(2^{p-1} - 1) + 1$$

By Fermat's little theorem the two factors are congruent to 2 and 0 mod  $p$  respectively, so:

$$\begin{aligned} A &\equiv 2(2^{p-1} - 1) + 1 \pmod{p^2} \\ &= 2^p - 1 \end{aligned}$$

(b) If  $i = 1$  then Granville's theorem can be used to show that:

$$A \equiv p + \frac{1}{2} \sum_{l=0}^{(p-1)/2} (-1)^l \binom{p+1}{l} \binom{p-2}{2l-1} \pmod{p^2}$$

Since the 1st binomial coefficient is congruent to 0 mod  $p$  for  $1 < l < p$ , we can simplify the 2nd binomial coefficient mod  $p$  via the congruence:

$$(1+x)^{p-2} \equiv (1+x^p)(1+x)^{-2} = (1+x^p) \sum_{k=0}^{\infty} (-1)^k (k+1) x^k \pmod{p}$$

and, subtracting a correction factor, obtain:

$$A \equiv \frac{1}{2}p - \sum_{l=0}^{(p-1)/2} (-1)^l \binom{p+1}{l} \cdot l \pmod{p^2}$$

By the identity  $\sum_{j=0}^k (-1)^j \binom{n}{j} j = (-1)^k \binom{n-2}{k-1} n$  (proved by writing  $\binom{n}{j} = \binom{n-1}{j-1} \frac{n}{j}$  and then applying the earlier cited identity  $\sum_{j=0}^k (-1)^j \binom{n}{j} = (-1)^k \binom{n-1}{k}$ ) and by the identity  $\binom{n-2}{k-1} = \binom{n-2}{k} \frac{k}{n-k-1}$ :

$$A \equiv \frac{1}{2}p - (-1)^{(p-1)/2} \binom{p-1}{(p-1)/2} \cdot \frac{p^2-1}{p+1} \pmod{p^2}$$

By Morley's congruence:

$$A \equiv \frac{1}{2}p + 2^{2(p-1)}(1-p) \pmod{p^2}$$

And again since  $2^{2(p-1)} \equiv 2^p - 1 \pmod{p^2}$ :

$$A \equiv 2^p - 1 - \frac{1}{2}p \pmod{p^2} \quad \square$$

PROOF OF PART (2) OF COROLLARY 13. If  $(m, m') = (p^{j-1} - 3, p^j - p^{j-1} + p^i - 5)$  then the coefficient of  $x_1^m x_2^{m'}$  is the  $k = \frac{1}{2}(p^{j-1} + 1)$  summand in Proposition 12. It is not difficult to show that this summand is congruent mod  $p^2$  to:

$$\begin{aligned} 4n_f^{(p^j+p^i)/2-8} & [(1-p) + \delta_{i=j-1} + 0 + 1 \\ & + B \\ & - (2^p - \delta_{i=j-1}) - (-\frac{1}{4}p^i) \\ & - 0 - (2 - \delta_{i=j-1}) \\ & - \frac{1}{4}p^i - 3\delta_{i=j-1}] \end{aligned}$$

where:

$$B = \frac{1}{2} \sum_{l=0}^{(p^{j-1}+1)/2} (-1)^l \binom{p^i+1}{l} \binom{p^j-p^i-2}{p^{j-1}-2l+1}$$

Due to tidy cancellations, *all that remains is to show that*  $B \equiv 2^p - p \pmod{p^2}$ .

(a) If  $i > 1$  then the above stated fact about  $\binom{p^i+1}{l}$  can be used to show that:

$$B \equiv \frac{1}{2} \sum_{r=0}^p (-1)^r \binom{p}{r} \binom{p^j-p^i-2}{p^{j-1}-2rp^{i-1}+1} \pmod{p^2}$$

The 1st binomial coefficient is congruent to 0 mod  $p$  for  $0 < r < p$ . The 2nd binomial coefficient is congruent to 0 mod  $p$  if  $0 < r < \frac{1}{2}(p+1)$  and congruent to  $-2 \pmod{p}$  if  $\frac{1}{2}(p+1) \leq r < p$ . So:

$$B \equiv \frac{1}{2} \binom{p^j-p^i-2}{p^{j-1}+1} - \sum_{r=(p+1)/2}^{p-1} (-1)^r \binom{p}{r} - \frac{1}{2} \binom{p^j-p^i-2}{p^{j-1}-2p^i+1} \pmod{p^2}$$

Granville's & Wolstenholme's theorems can be used to simplify the first and last terms mod  $p^2$  while the identity  $\sum_{j=0}^k (-1)^j \binom{n}{j} = (-1)^k \binom{n-1}{k}$  can be used to simplify the summation, yielding:

$$\begin{aligned} B \equiv (1 + \delta_{i=j-1} - p\delta_{i \neq j-2}) & - 1 + (-1)^{(p-1)/2} \binom{p-1}{(p-1)/2} - (\delta_{i=j-1} + p\delta_{i=j-2} - 1) \\ & \pmod{p^2} \end{aligned}$$

By Morley's congruence:

$$B \equiv 2^{2(p-1)} + 1 - p \pmod{p^2}$$

And again since  $2^{2(p-1)} \equiv 2^p - 1 \pmod{p^2}$ :

$$B \equiv 2^p - p \pmod{p^2}$$

(b) If  $i = 1$  then:

$$B = \frac{1}{2} \sum_{l=0}^{p+1} (-1)^l \binom{p+1}{l} \binom{p^j - p - 2}{p^{j-1} - 2l + 1}$$

Granville's theorem can be used to show that:

$$B \equiv \frac{1}{2}(p+2) + \frac{1}{2} \sum_{l=0}^{(p-1)/2} (-1)^l \binom{p+1}{l} \binom{p-2}{2l-1} \pmod{p^2}$$

This summation appeared above in the PROOF OF PART (1) OF COROLLARY 13, part (b). In fact  $B \equiv A + 1 - \frac{1}{2}p \pmod{p^2}$ . Since we concluded that  $A \equiv 2^p - 1 - \frac{1}{2}p^i \pmod{p^2}$ , it follows that:

$$B \equiv 2^p - p \pmod{p^2}$$

□

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